Topic – Linear Differential [illegible] with Constant Co-efficients (continued)

Class – B.Sc Part-II Date –

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Study Material –

Case III – Auxiliary Equation having imaginary roots :–

Let $\alpha \pm i\beta$ be the imaginary roots of the 2nd order

then the General Solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} \cdot c_2 e^{-i\beta x}$$

$$= e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x})$$

$$\left[\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ \therefore e^{i\beta x} = \cos\beta x + i\sin\beta x \\ e^{-i\beta x} = \cos\beta - i\sin\beta x \end{array} \right]$$

$$= e^{\alpha x} [c_1 (\cos\beta x + i\sin\beta x) + c_2 (\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x} [(c_1 + c_2)\cos\beta x + (c_1 - c_2) i\sin\beta x]$$

Ex :→ Solve

$$\frac{d^4y}{dx^4} + m^4 y = 0$$

olution
The Given Equation may be written as

$$(D^4+m^4)y=0$$

Auxiliary Equation

$$D^4+m^4=0$$

use. $[a^2+b^2=(a+b)^2-2ab]$

$$\Rightarrow (D^2+m^2)^2-2D^2m^2=0$$

$$\Rightarrow (D^2+m^2)^2-(\sqrt{2}Dm)^2=0$$

$$\Rightarrow (D^2+m^2+\sqrt{2}Dm)(D^2+m^2-\sqrt{2}Dm)=0$$

$$\Rightarrow D^2+\sqrt{2}Dm+m^2=0 \quad \text{and} \quad D^2-\sqrt{2}Dm+m^2=0$$

$$\Rightarrow D=\frac{-\sqrt{2}m\pm\sqrt{2m^2-4m^2}}{2} \qquad D=\frac{\sqrt{2}m\pm\sqrt{2-4m^2}}{2}$$

$$=\frac{-\sqrt{2}m\pm\sqrt{-2m^2}}{2} \qquad =\frac{\sqrt{2}m\pm\sqrt{2}mi}{2}$$

$$=\frac{-\sqrt{2}m\pm\sqrt{2}mi}{2} \qquad =\frac{m}{\sqrt{2}}\pm\frac{m}{\sqrt{2}}i$$

$$=-\frac{m}{\sqrt{2}}\pm\frac{mi}{\sqrt{2}}$$

$$\therefore y=e^{-\frac{mx}{\sqrt{2}}}\left[c_1\cos\frac{mx}{\sqrt{2}}+c_2\sin\frac{mx}{\sqrt{2}}\right]+e^{\frac{mx}{\sqrt{2}}}\left[c_3\cos\frac{mx}{\sqrt{2}}+c_4\sin\frac{mx}{\sqrt{2}}\right)$$

Ex:→ $(D^4+5D^2+6)y=0$

The Auxiliary equation is

$$D^4+5D^2+6=0 \qquad [\ \text{say } D'$$

This is a quadratic Equation in D^2

therefore $D^2=\frac{-5\pm\sqrt{25-24}}{2}=\frac{-5\pm1}{2}=-2,-3$

$\therefore D^2+2=0 \quad D^2+3=0 \Rightarrow D=\pm\sqrt{2}i \quad D=\pm\sqrt{3}i$

Hence the Solution

$$y = C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x + C_3 \cos\sqrt{2}x + C_4 \sin\sqrt{2}x$$

Ex: → Solve.

$$(D^4 - D^3 - D + 1)y = 0$$

Auxiliary Equation

$$D^4 - D^3 - D + 1 = 0$$

$$\Rightarrow D^3(D-1) - (D-1) = 0$$

$$\Rightarrow (D-1)(D^3 - 1) = 0$$

$$\Rightarrow (D-1)(D-1)(D^2 + D + 1) = 0$$

$$\therefore D = 1, 1, \quad D = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

Hence the Solution

$$y = (C_1 + C_2 x)e^{x} + e^{-\frac{1}{2}x}\left(C_3 \cos\frac{\sqrt{3}}{2}x + C_4 \sin\frac{\sqrt{3}}{2}x\right)$$